Equivariant Syzygies of the Ideal of 2×2 Permanents of a $2 \times n$ Matrix

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Texas A&M Geometry Seminar, April 25, 2025

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Notation

- k a field of characteristic 0.
- $V = k$-vector space of dimension m.
- $W = k$-vector space of dimension n.
- $S = \mathrm{Sym}(V \otimes W)$, the polynomial ring over k with variables the entries of an $m \times n$ matrix:

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}$$

Syzygy Example

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

Let I be the ideal of 2×2 minors of X.

$$I = (x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2)$$

We are interested in the minimal free resolution of S/I.

$$0 \leftarrow S \xleftarrow{\begin{pmatrix} x_1y_2 - x_2y_1 & x_1y_3 - x_3y_1 & x_2y_3 - x_3y_2 \end{pmatrix}} S^3 \xleftarrow{\begin{pmatrix} -x_3 & y_3 \\ x_2 & -y_2 \\ -x_1 & y_1 \end{pmatrix}} S^2 \leftarrow 0$$

Betti Tables

	0	1	2
0	1	-	-
1	-	3	2

The **Betti Table** summarizes important information about the resolution. The **Betti number** in column p and row q is

	$\dots$	p	$\dots$
$\vdots$	$\dots$	$\dots$	$\dots$
q	$\dots$	$\dim_k \operatorname{Tor}_p^S(S/I, k)_{p+q}$	$\dots$
$\vdots$	$\dots$	$\dots$	$\dots$

Determinantal Ideals

One natural ideal associated to X is the ideal of $d \times d$ minors of X. Its corresponding variety parameterizes $m \times n$ matrices of rank $< d$.

Lascoux ('78) determined their minimal free resolutions, which are a generalization of the Eagon-Northcott complex.

Example: Two Row Matrix

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ y_1 & y_2 & y_3 & \dots & y_n \end{pmatrix}$$

The ideal of 2×2 minors is

$$(x_i y_j - x_j y_i : i < j)$$

Its resolution is given by the Eagon-Northcott Complex, producing Betti table

	1	2	3	4	...	$n-1$
1	$\binom{n}{2}$	$2\binom{n}{3}$	$3\binom{n}{4}$	$4\binom{n}{5}$	...	$(n-1)\binom{n}{n}$

Group Actions

There is a natural $\mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$-action on S by

$$\mathrm{GL}_m(k) \curvearrowright \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \curvearrowleft \mathrm{GL}_n(k)$$

Equivariant Betti Tables

When an ideal is stable under the action of a subgroup $G \leq \mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$, the k-vector spaces $\mathrm{Tor}_i^S(S/I, k)$ become G-representations.
We can thus produce a *G-equivariant Betti table*

The ideal of $d \times d$ minors is stable under $\mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$. The equivariant Betti table for the 2×2 minors of a $2 \times n$ matrix is

	1	2	…	$n-1$
1	$\mathbb{S}_{(0)} V \otimes \mathbb{S}_{(1^2)} W$	$\mathbb{S}_{(1)} V \otimes \mathbb{S}_{(1^3)} W$	…	$\mathbb{S}_{(n-2)} V \otimes \mathbb{S}_{(1^n)} W$

Permanents of a Matrix

The permanent of a square matrix is the determinant without any negative signs.

$$\operatorname{perm}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad + bc$$

$$\operatorname{perm}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh + afh + bdi + ceg$$

$$A = (a_{ij}), \quad \operatorname{perm}(A) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

Permanental Ideals

The ideal of $d \times d$ subpermanents of X remains mysterious.

Areas of Interest

- Graph Theory
- Geometric Complexity Theory

Known Results

- Labenbacher, Swanson '00 - Gröbner basis and primary decomposition when $d = 2$.
- Kirkup '08 - Minimal primes when $d = 3$
- Efremenko, Landsberg, Schenck, Weyman '18 - First linear strand of resolution when X is square.
- Boralevi, Carlini, Michałek, Ventura - '24 - Some results on codimension
- Gesmundo, Huang, Schenck, Weyman '24 - Betti table for 2×2 permanents of $2 \times n$ matrix.

Betti Tables for Permanents of $2 \times n$ matrix

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ y_1 & y_2 & y_3 & \dots & y_n \end{pmatrix}$$

$$I = (x_i y_j + x_j y_i : i < j)$$

	1	2	3	4	5
1	6	-	-	-	-
2	-	22	24	6	-
3	-	-	4	8	3

n=4

	1	2	3
1	3	-	-
2	-	3	-
3	-	-	1

n=3

	1	2	3	4	5	6	7
1	10	-	-	-	-	-	-
2	-	80	170	150	60	10	-
3	-	-	10	40	55	30	6

n=5

Solution

Theorem (Gesmundo, Huang, Schenck, Weyman '24, Z. '25)

The following is the Betti table for the ideal of 2×2 permanents of a $2 \times n$ matrix.

	1	p
1	$\binom{n}{2}$	-
2	-	$\binom{n}{2}\binom{2n-4}{p} - \binom{n}{p+2}(2^{p+2}-2)$
3	-	$\binom{n}{2}\binom{2n-4}{p-2} - n\binom{2n-2}{p} + \binom{2n}{p+2} - \binom{n}{p+2}2^{p+2}$

My Problem

$$I = (x_i y_j + x_j y_i : i \neq j)$$

Issue: This result doesn't respect natural group actions on the ideal/resolution.

Goal: Determine the equivariant Betti table of I under its natural group actions.

Group of Monomial Matrices

The subgroup of **monomial matrices** of $\mathrm{GL}_n(k)$ has one non-zero entry in each row and column

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix} \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}$$

Group theoretically, this is

$$(\mathfrak{S}_n \ltimes (k^*)^n) =: G_n$$

Group Actions on Permanents

The ideal of permanents is invariant under

$$G_m \times G_n = (\mathfrak{S}_m \ltimes (k^*)^m) \times (\mathfrak{S}_n \ltimes (k^*)^n) = G_m \times G_n$$

Symmetric group action: $\mathfrak{S}_m$ permutes rows of X, while $\mathfrak{S}_n$ permutes columns.

Torus action: The ideal has a $\mathbb{Z}^m \times \mathbb{Z}^n$ grading where x_{ij} has multidegree $(0, \ldots, 1_i, \ldots, 0) \times (0, \ldots, 1_j, \ldots, 0)$.

Example: Action on Generators

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

$$I = (x_1 y_2 + x_2 y_1, x_1 y_3 + x_3 y_1, x_2 y_3 + x_3 y_1)$$

$\mathrm{Tor}_1(S/I, k)$ is the vector space spanned by the generators of I.

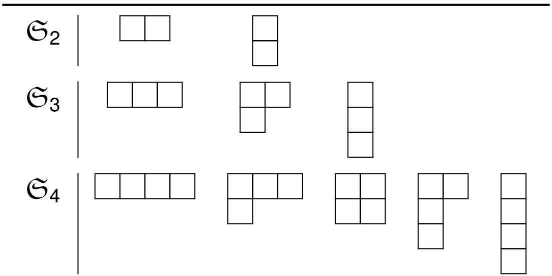
The $\mathfrak{S}_2 \times \mathfrak{S}_3$-action fixes the vector space.

The generators live in $\mathbb{Z}^2 \times \mathbb{Z}^3$ multidegrees

$$(1,1) \times (1,1,0), (1,1) \times (1,0,1), (1,1) \times (0,1,1)$$

Complex Symmetric Group Representations

Now let $k = \mathbb{C}$. Irreducible Representations of $\mathfrak{S}_n$ are parameterized by **Young diagrams**: boxes arranged in rows of decreasing size.



Basic Symmetric Representations

- Trivial representation: Young diagram is a row $\ydiagram{4}$. Action does nothing.

$$\sigma(v) \rightsquigarrow v$$

- Sign representation: Young diagram is a column $\ydiagram{1,1,1,1}$.

$$\sigma(v) \rightsquigarrow \operatorname{sgn}(\sigma) v$$

- Young diagram a hook $\ydiagram{3,1,1}, \ydiagram{2,1,1,1,1}, \ldots$ are the exterior powers of the "standard representation" $\ydiagram{3,1}$

Multigraded Symmetric Representations

Irreducible representations of G_n are induced representations from the Young subgroup stabilizing a multidegree **a**.

$$\mathrm{Ind}_{\mathrm{Stab}(\mathbf{a})}^{\mathfrak{S}_n}(\rho_1 \boxtimes \cdots \boxtimes \rho_\ell)_{\mathbf{a}} =: (\rho_1, \ldots, \rho_\ell)_{\mathbf{a}}$$

Example

$$I = (x_1y_2 + x_2y_1, x_1y_3 + x_3y_1, x_2y_3 + x_3y_1)$$

To write $G_2 \times G_3$ representation of $\mathrm{Tor}_1(S/I, \mathbb{C})$:

1. Basis of fixed $\mathbb{Z}^2 \times \mathbb{Z}^3$-Multidegree: $(1,1) \times (1,1,0) \to \{x_1y_2 + x_2y_1\}$
2. $\mathfrak{S}_2 \times \mathfrak{S}_3$-Stabilizer of multidegree: $\mathfrak{S}_2 \times (\mathfrak{S}_2 \times \mathfrak{S}_1)$
3. Representation on each component: Trivial
4. $G_2 \times G_3$-representation: The induced representation

$$(\square\square)_{(1,1)} \otimes (\square\square, \square)_{(1,1,0)}$$

By contrast, the representation corresponding to the space of minors $x_iy_j - x_jy_i$ is

$$\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)_{(1,1)} \otimes \left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\right)_{(1,1,0)}$$

Main Theorem

Theorem (Z. ’25)

The following is G_n-equivariant Betti table of the ideal $(x_iy_j + x_jy_i : i < j)$ over $\mathbb{C}$:

	1	p
1	$(\ydiagram{2})_{(1^2,0^{n-2})}$	-
2	-	$\sum\limits_{2a+b=p+2} \sum_{c=0}^{b-2}\Big(\big(\underbrace{\ydiagram{4}}_{a}, (\ydiagram{1}, \underbrace{\ydiagram{1,1,1}}_{b-2-c}, \underbrace{\ydiagram{1,1,1}}_{c})\big)_{(2^a,1^b,0^{n-a-b})} - \big((\underbrace{\ydiagram{1,1,1}}_{b+1-c}, \underbrace{\ydiagram{1,1,1}}_{c+1})\big)_{(1^b,0^{n-b})}\Big)$
3	-	$\sum\limits_{2a+b=p+3} \sum_{c=0}^{b}\big(\underbrace{\ydiagram{4,1,1}}_{a-2,1,1}, (\underbrace{\ydiagram{1,1,1}}_{b-c}, \underbrace{\ydiagram{1,1,1}}_{c})\big)_{(2^a,1^b,0^{n-a-b})}$

Proof Idea

Idea: Use a related monomial ideal.

GHSW'24 — Use the initial ideal.

Downside: Doesn't respect group action

Proof Outline

Our strategy — Use the filtration

$$(x_iy_j + x_jy_i : i \neq j) \subset (x_iy_j : i \neq j) \subset (x_iy_j : \text{any } i, j)$$

- Inclusion maps respect $G_2 \times G_n$ action.
- Apply $\mathrm{Hom}_S(-, \mathbb{C})$ to short exact sequences.
- Maps respect $G_2 \times G_n$ action **and** $\mathrm{Ext}_S(\mathbb{C}, \mathbb{C})$-module structure.

Ingredients

- Two short exact sequences

$$0 \to (x_iy_j + x_jy_i : i \neq j) \hookrightarrow (x_iy_j : i \neq j) \xrightarrow{\varphi} \bigoplus_{i,j} \frac{k[x_i, x_j, y_i, y_j]}{(x_iy_j + x_jy_i)} \overline{x_iy_j - x_jy_i} \to 0$$

$$0 \to (x_iy_j : i \neq j) \hookrightarrow (x_iy_j : \text{any } i, j) \xrightarrow{\psi} \bigoplus_{i} k[x_i, y_i]\overline{x_iy_i} \to 0$$

- The maps $\operatorname{Ext}(\varphi, \mathbb{C}), \operatorname{Ext}(\psi, \mathbb{C})$ are either injective, surjective, or 0 at each multidegree.
- To get representations: Analyze $G_2 \times G_n$-representations of $\operatorname{Ext}_S(\mathbb{C}, \mathbb{C}) \cong \bigwedge(V \otimes W)$.

Conclusion

Upshot: Respecting the group action made for a stronger and more elegant result.

Future directions

Constructing Resolutions?

Larger matrices?

Bigger permanents?

Thank You!