A History of Calculus for Instructors

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Goal

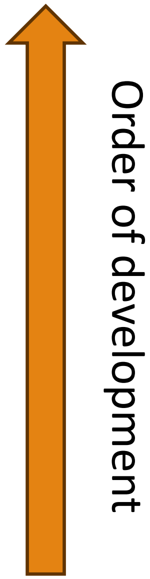
Understand how the concepts of calculus, as taught in a standard Calculus 1 course, came to be.

What is Calculus?

Calculus is the study of instantaneous rates of change and areas under curves.

Concepts required:

- Real number line
- Function
- Limit
- Continuity
- Derivative
- Integral



And More…

- Graphical representation
- Algebraic notation
- Intuition about infinite/infinitesimal processes
- Infinite series

Outline

We will follow a chronological development

I. Antiquity: Origins of Math (500BC-300AD)
II. Late Antiquity + Medieval: Changing Perspectives (300-1500)
III. Renaissance: Developments in Algebra and Analysis (1500-1665)
IV. Birth of Calculus: Newton and Leibniz (1665-1687)
V. Foundations of Calculus: Limits and the Calculus of Today (1687-1872+)

Remarks

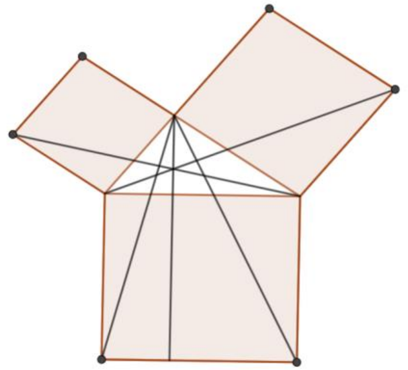
- My information is from secondary sources
- People cited are often exemplary, not the first to discover idea
- Many important people left out

Antiquity: Origins of Math

500BC-300AD

DEVELOPMENT OF GEOMETRY. SEPARATION OF NUMBER, MOTION, AND INFINITE(SIMAL) FROM GEOMETRY. FINDING AREAS BY "EXHAUSTION"

Beginnings of Geometry



The Pythagorean Theorem

First proofs and general theorems come from ancient Greece

Pythagoreans (~500BC)

Theorem (Irrationals): There are distinct line segments A and B such that there is no unit line segment small enough such that A and B are collections of units. (e.g. diagonal and side of unit square)

=> "number" and magnitude are separate

Area found by comparing shapes

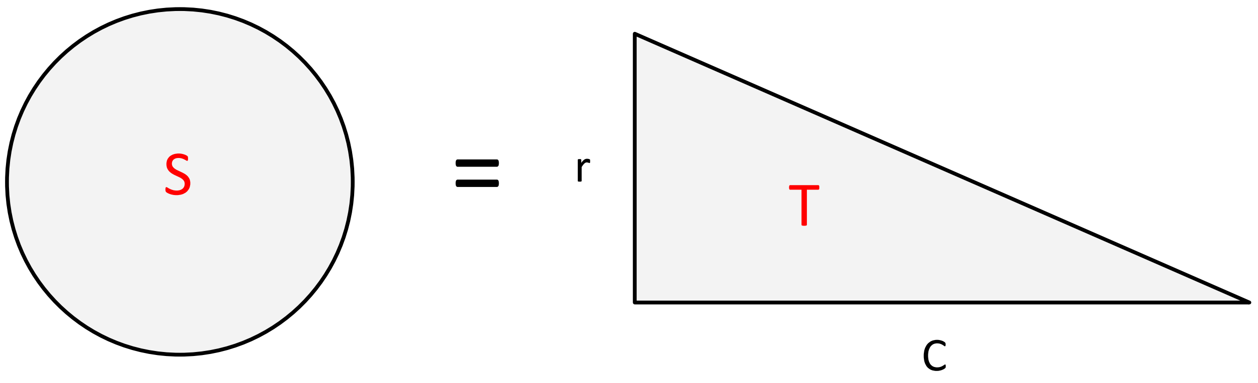
Advanced Area Finding: Method of Exhaustion

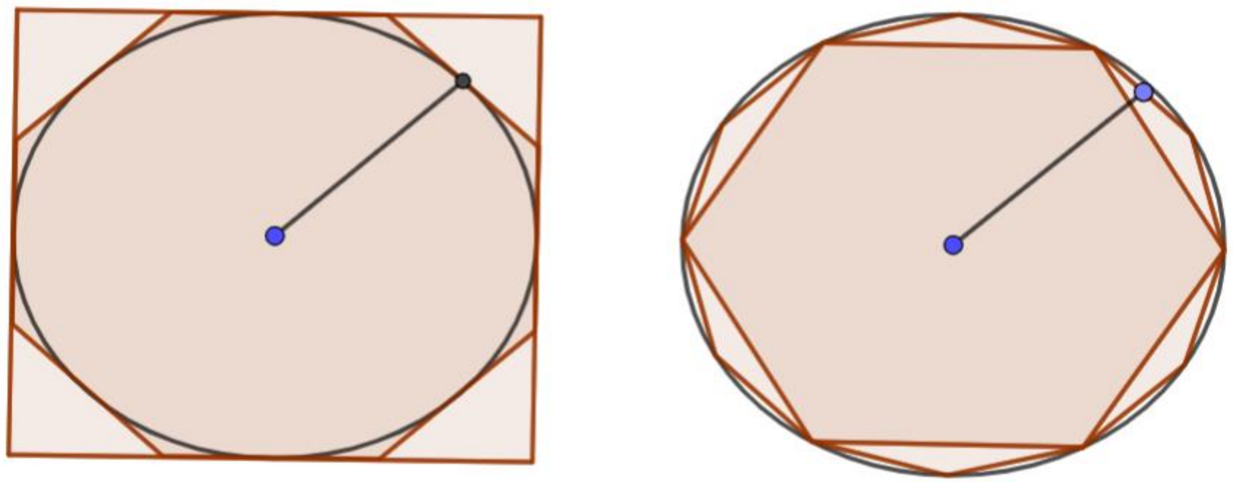
"Method of Exhaustion" (MOE): Claim: A shape S has the same area as a shape T

1) Assume T is bigger than S
2) Inscribe a figure P (polygon) in S
3) Modify P by a finite process, obtaining inscribed figures P1,P2,…
4) Show that Pn can be as close to T as we like (Contradiction)
5) Repeat: Assume T is smaller than S, circumscribe, modify, contradiction

Eudoxus (423-347BC)

MOE Example:





Lack of Infinitude

"Atomism", the idea that matter can be broken into tiny, indivisible parts

Democritus (460-370BC) found areas of pyramids, cones using indivisibles, but such methods fell out of favor because…

Zeno's (495-430BC) Paradoxes: No such thing as instantaneous (or any…) motion

Future Influences

Euclid's (~300BC) *Elements*: A text for all time

- Rigorous axiomatic mathematics
- Primarily geometry, number theory

Archimedes (287-212BC)

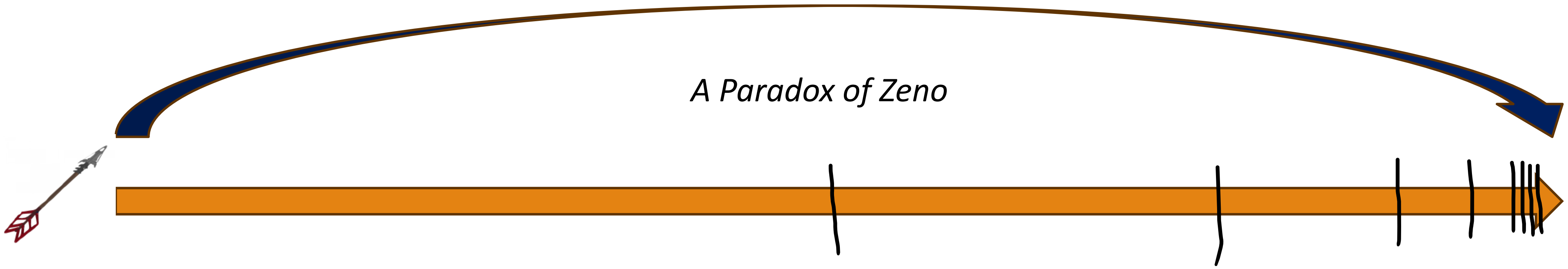
- Computed Volumes, surface areas, centers of gravity of spheres, cones, parabolas, etc. using MOE
- Found tangent to a spiral

Future Influences, Continued

Aristotle (384-322BC)

- Instantaneous motion is possible, though not quantitative
- No actual indivisibles, but infinite exists "potentially" and can be approached

Apollonius (15-100AD) *Conics*, Diophantus (214-298AD) Algebra?



Late Antiquity and Medieval Times: Changing Perspectives

300-1500

ORIGINS OF ALGEBRA. NUMERALS. GRAPHICAL REPRESENTATION. COMFORT WITH THE INFINITE

Numerals

Decimal system developed in India by the 6th Century, as recorded in the *Aryabhatiya* of Aryabhata. It combined:

i. Base 10

ii. Positional notation

iii. Symbol for each numeral 1-9 (later 0 in the 9th century)

Origins of Algebra

“al-jabr” – completion – subtract one side of the equation, add to the other

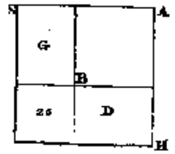
“muqabalah” – reduction – cancel terms on opposite sides

Al-Khwarizmi (780-850)

- *Hisob al-jabr* responsible for algebraic processes and adoption of Hindu numerals

An English translation of Al-Khwarizmi’s *Algebra*

the first quadrate, which is the square, and the two quadrangles on its sides, which are the ten roots, make together thirty-nine. In order to complete the great quadrate, there wants only a square of five multiplied by five, or twenty-five. This we add to thirty-nine, in order to complete the great square S H. The sum is sixty-four. We extract its root, eight, which is one of the sides of the great quadrangle. By subtracting from this the same quantity which we have before added, namely five, we obtain three as the remainder. This is the side of the quadrangle A B, which represents the square; it is the root of this square, and the square itself is nine. This is the figure:—



Demonstration of the Case: “a Square and twenty-one Dirhems are equal to ten Roots.”

We represent the square by a quadrate A D, the length of whose side we do not know. To this we join a parallelogram, the breadth of which is equal to one of the sides of the quadrate A D, such as the side H N. This parallelogram is H B. The length of the two

Source: Wikipedia

Graphs and Infinities

Translation of Islamic texts in 12th century led to Greek (esp. Aristotelian) revival and adoption of Islamic math in the Latin West

Infinite Q's:

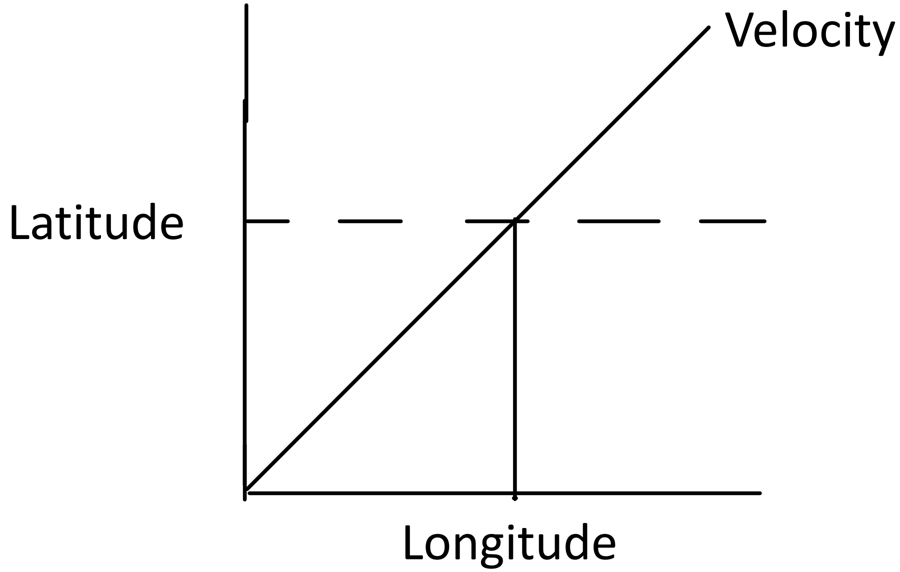
- Is infinity just potential or actual?
- Are the infinite magnitudes or collections?
- Does a line just contain points or consist of them?

Aristotelian "''forms''": inherent qualities that change continuously (e.g. speed, heat)

Graphs and Infinities

Oresme (1325-1382)

- “Latitude of Forms”-Graphical representation of quantities of certain qualities, e.g. motion over time
- Infinite series – **Theorem:** The harmonic series diverges



Renaissance: Developments in Algebra and Analysis

1500-1665

ALGEBRA IN EARNEST. NEW NUMBERS AND NOTATION. EXTENDING THE MOE. INFINITESIMAL METHODS. MARRYING ALGEBRA AND GEOMETRY. THE RULES OF CALCULUS.

Development of Algebra

Solving of the cubic (Tartaglia + Cardano) and quartic (Ferrari)

Vieta (1540-1603): Represent all polynomials of a given degree with one equation

- Letters for knowns and unknowns

$$ax^2 + bx + c$$

Comfort with rationals, negatives, and complex numbers growing

Guess That Notation!

p	1
1	2

3/2 Power

R)^2.

Square root

$\bar{p}$

plus

$\bar{m}$

minus
(moins)

cosa

Unknown
(thing)

∝

Equals

⑤ ② ①
4 + 3 - 8

$4x^5 + 3x^2 - 8x$

Extending the MOE

Using lemmas as a shortcut for the MOE

Using infinitesimal methods in place of the MOE

Cavalieri (1598-1647)

- Cloth = sum of threads, Book = sum of pagese
- **Theorem**: Power rule $\int x^n$ (Later: Toricelli for rational powers)

Kepler (1598-1647)

- Volumes of 92 shapes beyond Archimedes
- Optimization

Analytic Geometry: Marrying Geometry and Algebra

"Analytic Geometry": Associate each curve with an equation that implies the properties of the curve.

- Graphical representation and algebraic equations intertwined.

Descartes' (1596-1650) "La geometrie"

Fermat (1607-1665)

- Used infinitesimals algebraically: Small change E -> calculate -> Let E = 0
- Area under $y^2=x^3$ using tangent lines, using infinitesimals algebraically and geometrically

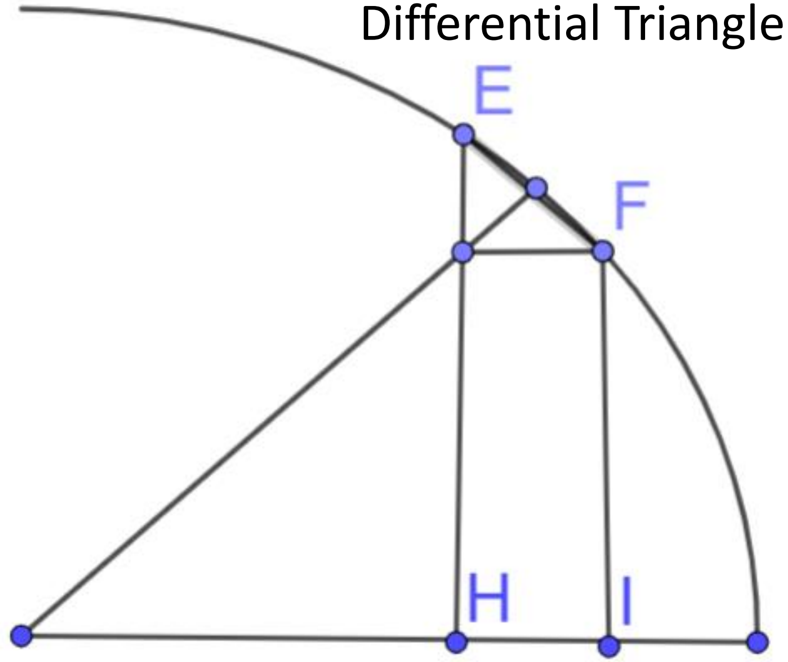
Inspirations to Newton and Leibniz

Pascal (1623-1662)
- Differential triangle
- “Pascal’s” Triangle
- **Theorem**: Integration by Parts

Wallis (1616-1703)
- Manipulating infinity algebraically, computed series
- Introduced symbol ∞

Barrow (1616-1677) Newton’s mentor
- Derivative and integral rules
- **Fundamental Theorem of Calculus**

Differential Triangle



Birth of Calculus: Newton and Leibniz

1665-1687

UNITING THE RESULTS OF CALCULUS INTO A DISTINCT FIELD AND GENERAL METHOD.

Newton (1642-1726)

Inspired by **Binomial Theorem**

3 treatises explaining the method of calculus 1665,1671,1676

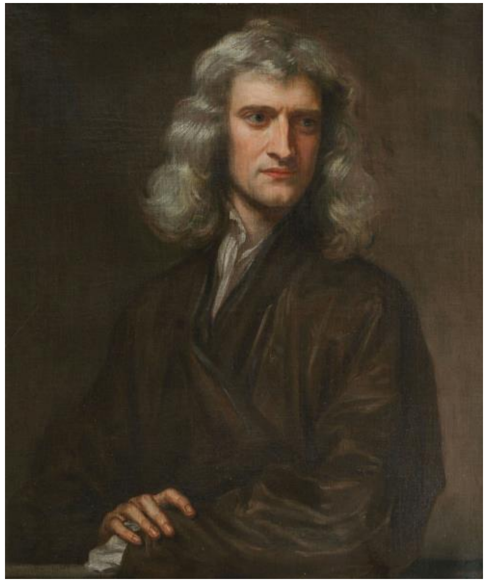
- + *Principia* 1687

"Fluxions": Rate of generation of a quantity

Rate of generation of a quantity: $\dot{x}$

Quantity generated: x

The quantity for which x is a fluxion (i.e. "fluent"): $\acute{x}$



Source: Wikipedia

Example: Power rule for derivatives

Area = $n/(n+m)\ x^{(m+n)/n}$, Curve = ???

Small change in x -> x + o

Area + oy = $n/(n+m)\ (x+o)^{(m+n)/n} = n/(n+m)\ x^{(m+n)/n} + x^{m/n}\ o + o^2(\text{stuff})$

$y = x^{m/n} + o(\text{stuff})$

$y = x^{m/n}$

Leibniz (1646-1716)

Inspired by Paschal’s Triangle

Notation: dx and ∫ydx

Developed 1672-1676, Published 1684

“Differential and Summatory Calculus”: Differentials are fundamental

$$1$$
$$\frac{1}{2} \quad \frac{1}{2}$$
$$\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{3}$$
$$\frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{4}$$
$$\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{30} \quad \frac{1}{20} \quad \frac{1}{5}$$
$$\frac{1}{6} \quad \frac{1}{30} \quad \frac{1}{60} \quad \frac{1}{60} \quad \frac{1}{30} \quad \frac{1}{6}$$
$$\frac{1}{7} \quad \frac{1}{42} \quad \frac{1}{105} \quad \frac{1}{140} \quad \frac{1}{105} \quad \frac{1}{42} \quad \frac{1}{7}$$
$$\frac{1}{8} \quad \frac{1}{56} \quad \frac{1}{168} \quad \frac{1}{280} \quad \frac{1}{280} \quad \frac{1}{168} \quad \frac{1}{56} \quad \frac{1}{8}$$
$$\vdots \qquad \vdots \qquad \vdots$$

Source: Wikipedia

Source: Wikipedia

Calculus is invented. What's left?

- Integral ✓
- Derivative ✓
- Continuity
- Limit
- Function
- Real number line

} The Foundations!

Foundations of Calculus: Limits and the Calculus of Today

1687-1872+

ESTABLISHING THE LIMIT. CONTINUITY. CONVERGENT SERIES. EPSILON-DELTA. THE REAL NUMBER LINE. +LOGICAL INFINITESIMALS.

What is the logical foundation of Calculus?

General Problems:

- How can we justify “cancelling zeros”? (Contradiction?)
- How do we understand higher order differentials?

Answer 1: Fluxions

- Problem: Relies on intuition about motion, which may not be correct or relevant

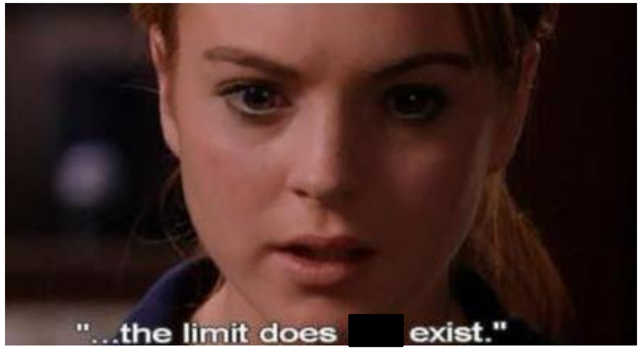
Answer 2: Differentials

- Problem: General confusion. Higher order differentials

Answer 3: Functions (advocated by Lagrange (1736-1813))

- Taylor series fundamental: f’(x) is just a coefficient
- Problem: Not every function has a Taylor series expansion

Answer 4: Limits



"One quantity is the limit of another if the second approaches the first nearer than any given quantity, so that the difference between them is absolutely inassignable" –D'Alembert (1717-1783)

Advocated by L'Hulier in 1787 in response to Lagrange

1804 textbook by Lacroix popularized limits and Leibniz notation

Problem: Alleged "Law of Continuity": "If a variable enjoys a property at all stages the limit will enjoy the same property" –L'Hulier

Relied on geometric intuition

Modern Definitions

Thanks to Cauchy (1789-1857)

- Continuous functions
- Infinitesimal is just a variable going to 0
- Orders of vanishing -> Higher order differentials make sense now
- Convergence of series

Lingering Problem: Number as length of line segment. Cauchy sequences should converge?

Real Number Line

Weierstrass (1815-1897)

- Established epsilon-delta definition of limit
- 1872 constructed continuous nowhere differentiable function (shoutouts to Bolzano)

1872 Weierstrass, Cantor, Meray, Heine, and Dedekind constructed real numbers

Example (Dedekind Cut):

That’s it!

The results of calculus have been laid on a solid logical foundation, and thus most histories of calculus end with the construction of the reals in 1872.

But what if there is another logical foundation for calculus???

+Infinitesimals work!

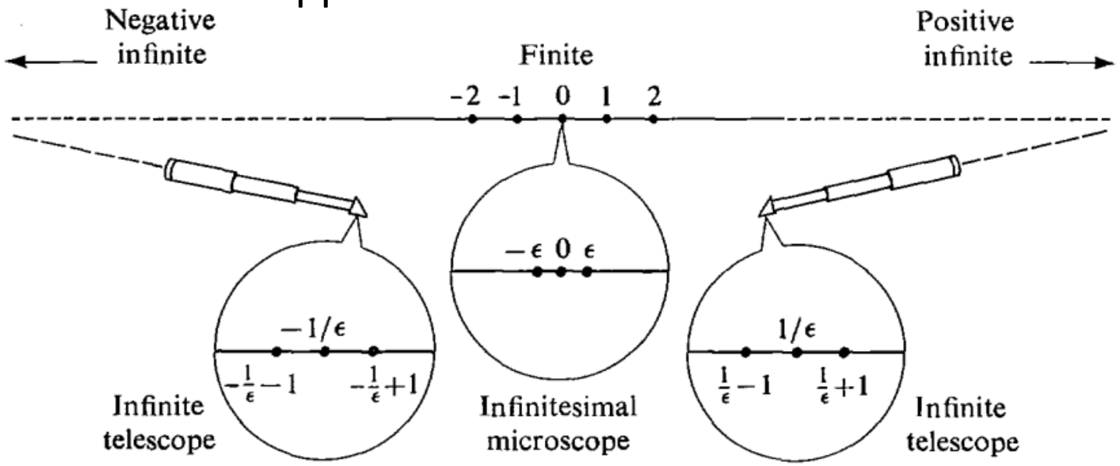
1934 Skolem introduces ultraproduct

1948 Hewitt constructs hyperreals (includes infinite and infinitesimal numbers)

1961 Robinson gives rigorous treatment of calculus using hyperreals

The Hyperreal Number Line

Credit: H. Jerome Keisler, in Elementary Calculus: An Infinitesimal Approach



Thank you!

Sources/Further Reading:

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