Monomial Cycles in Koszul Homology

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Introduction

Let k be a field, $Q = k[x_1, \dots, x_n]$, $I \subset Q$ a monomial ideal generated in degree ≥ 2, and $R = Q/I$. Understanding $\mathrm{Tor}^Q_\bullet(R,k)$ is a rich area of research in commutative algebra. It can be computed by taking the homology of the **Koszul complex** of R:

$$R \xleftarrow{\partial} \bigwedge^1 R^n \xleftarrow{\partial} \bigwedge^2 R^n \xleftarrow{\partial} \dots \xleftarrow{\partial} \bigwedge^n R^n \leftarrow 0$$

$$\partial(\bar{f} e_{i_1} \wedge \dots \wedge e_{i_d}) = \sum_{j=1}^{d} \overline{f x_{i_j}} e_{i_1} \wedge \dots \wedge \hat{e}_{i_j} \wedge \dots \wedge e_{i_d}$$

The exterior algebra structure on the Koszul complex extends to an algebra structure $\mathrm{Tor}^S_d(R,k) \cdot \mathrm{Tor}^S_{d'}(R,k) \subset \mathrm{Tor}^S_{d+d'}(R,k)$ given by, e.g. $[\bar{f} e_i][\bar{g} e_j] = [\overline{fg} e_i \wedge e_j]$
This algebra structure is important in the study of (infinite) resolutions over R.
When the product (+higher order products) is trivial, I is called **Golod**, and resolutions over R can be computed in terms of (finite) Q-resolutions. Our goal is to understand the vanishing of this product when f, g are monomials.
A **monomial cycle** of the Koszul complex is a cycle that can be written as $\bar{u} e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_d}$ where u is a monomial of Q. In other words, $u \in I : (x_{i_1}, \dots, x_{i_d})$. A product of monomial cycles is again a monomial cycle, so understanding when a monomial cycle is a boundary tells us when a product of such cycles is a boundary.

Main Theorem

Theorem. A monomial cycle $\bar{u} e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_d}$ is a boundary if and only if $u \in B_I^{n,\{i_1,\dots,i_d\}}$.

$B_I^{n,\{i_1,\dots,i_d\}}$ is an ideal we introduce called the boundary ideal. The combinatorics of full simplicial matroids give formulae for computing boundary ideals.

Full Simplicial Matroids

To any set of vectors over a field, we can associate a **matroid**, which keeps track of the subsets of those vectors which are (in)dependent. The dependent sets which are minimal with respect to inclusion are called **circuits**.
The **full simplicial matroid** S^n_d is a matroid associated to an n-simplex. The differential mapping d-faces to $(d-1)$-faces of the simplex is a matrix. The full simplicial matroid S^n_d is the matroid corresponding to the row vectors of that matrix.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \end{array} \\ \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{array}{c} \begin{array}{cccc} 123 & 124 & 134 & 234 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ 123 \\ 124 \\ 134 \\ 234 \end{array} \begin{array}{c} 1234 \\ \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \end{array}$$

	S^4_1	S^4_2	S^4_3
Circuits containing first row:	$\{1,2,3,4\}$	$\{12,13,14\}\{12,23,24\}$ $\{12,13,24,34\}\{12,14,23,34\}$	$\{123,124\}$ $\{123,134\}$ $\{123,234\}$

Figure: The matrices corresponding to the full simplicial matroids S^4_1, S^4_2, and S^4_3, respectively.

Boundary Ideals

The **boundary ideal** associated to a subset A of $\{1,2,\dots,n\}$ of cardinality d is

$$B_I^{n,A} := \bigcap_{\substack{C \text{ a circuit of } S^n_d \\ \text{such that } A \in C}} \sum_{B \in C} x_{B \setminus A}[I : x_{A \setminus B}]$$

where $x_T := \prod_{t \in T} x_t$. For example, the circuits of S^4_1, S^4_2, S^4_3 are on the bottom left. So,

$$\begin{aligned} B_I^{4,\{1\}} &= I + x_2[I : x_1] + x_3[I : x_1] + x_4[I : x_1] = I + (x_2, x_3, x_4)[I : x_1] \\ B_I^{4,\{1,2\}} &= (I + x_3[I : x_2] + x_4[I : x_2]) \leftrightsquigarrow \{12, 13, 14\} \\ &\cap (I + x_4[I : x_2] + x_3[I : x_1] + x_3x_4[I : x_1x_2]) \leftrightsquigarrow \{12, 14, 23, 34\} \\ &\cap (I + x_3[I : x_2] + x_4[I : x_1] + x_3x_4[I : x_1x_2]) \leftrightsquigarrow \{12, 13, 24, 34\} \\ &\cap (I + x_3[I : x_1] + x_4[I : x_1]) \leftrightsquigarrow \{12, 23, 24\} \\ &= I + (x_3, x_4)[I : (x_1, x_2)] + (x_3x_4)[I : x_1x_2] \cap (x_3[I : x_1] \cap x_4[I : x_2] + x_3[I : x_2] \cap x_4[I : x_1]) \\ B_I^{4,\{1,2,3\}} &= (I + x_4[I : x_3]) \cap (I + x_4[I : x_2]) \cap (I + x_4[I : x_1]) = I + x_4[I : (x_1, x_2, x_3)] \end{aligned}$$

Matroid-Boundary Ideal Dictionary

Matroid	Circuits	Boundary Ideal
S^n_1	$\{1,2,\dots,n\}$	$B_I^{n,\{1\}} = I + (x_2,\dots,x_n)[I : x_1]$
S^n_2	2-color graphs	formula for $B_I^{n,\{1,2\}}$
$S^n_3,\dots,S^n_{n-3}$	unknown, characteristic dependent [1] [4]	$B_I^{n,A}$ is characteristic dependent
S^n_{n-2}	complements of graph circuits	formula for $B_I^{n,\{1,\dots,n-2\}}$
S^n_{n-1}	subsets size 2	$B_I^{n,\{1,\dots,n-1\}} = I + x_n[I : (x_1,\dots,x_{n-1})]$
S^n_n	none	$B_I^{n,[n]} = I$

Figure: The circuits of the full simplicial matroid S^4_2 containing 12; they are the two-color graphs on four vertices where 1 and 2 are different colors. Alternatively, they are the complements of the circuits (in the sense of graphs) containing the edge 34.

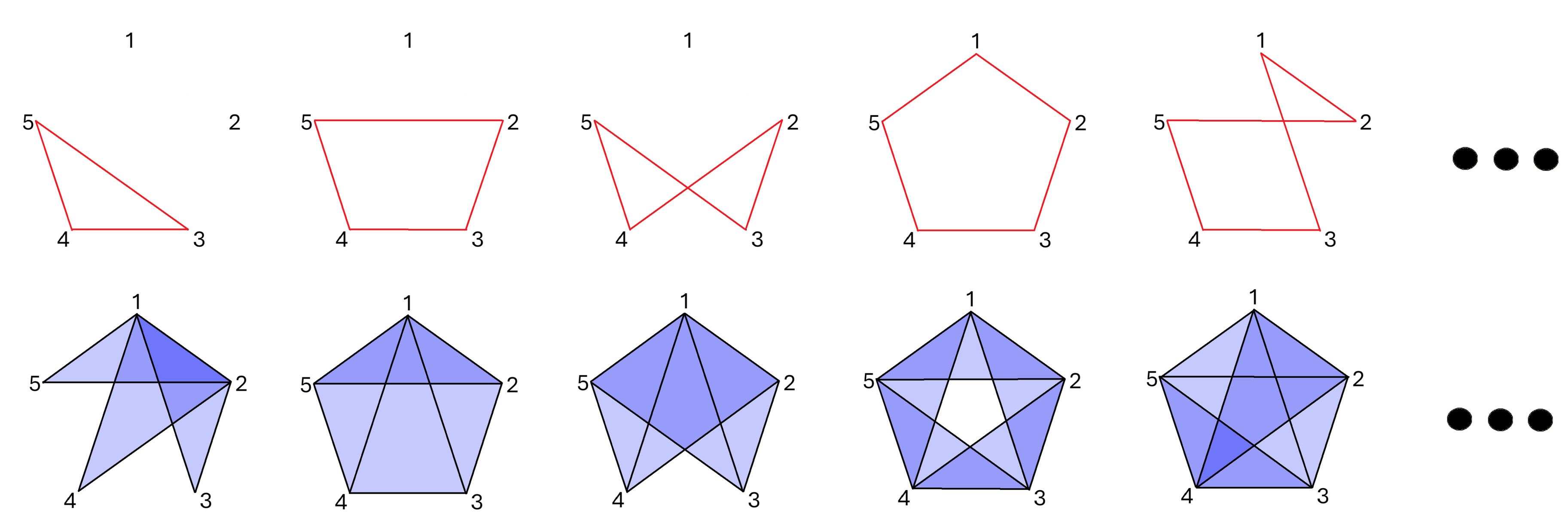

Figure: The circuits of the full simplicial matroid S^5_3 containing 123; they are the complements of the circuits (in the sense of graphs) on five vertices containing the edge 45.

Consequences

Theorem. Products of monomial cycles $[\bar{u} e_1 \wedge \dots \wedge e_{d'}][\bar{v} e_{d'+1} \wedge \dots \wedge e_d]$ vanish if and only if the following inclusion of ideals holds:

$$[I : (x_1, \dots, x_{d'})][I : (x_{d'+1}, \dots, x_d)] \subset B_I^{n,\{1,\dots,d\}}$$

In four or fewer variables, I is Golod if and only if all products of $\mathrm{Tor}^S_{\geq 1}(R,k)$ vanish. In fact, we prove that it suffices to check only products of monomial cycles. We thus prove the following theorem:

Theorem. I is a monomial Golod ideal in four variables if and only if the following inclusions hold under all permutations of the indices:

$$[I : x_1][I : (x_2, x_3, x_4)] \subset I$$

$$[I : (x_1, x_2)][I : (x_3, x_4)] \subset I$$

$$[I : x_1][I : (x_2, x_3)] \subset I + x_4[I : (x_1, x_2, x_3)]$$

$$[I : x_1][I : x_2] \subset I +$$
$$(x_3, x_4)[I : (x_1, x_2)] + (x_3x_4)[I : x_1x_2] \cap (x_3[I : x_1] \cap x_4[I : x_2] + x_3[I : x_2] \cap x_4[I : x_1])$$

This generalizes a classification for three or fewer variables in [2].

Questions

- Is there a general formula for the circuits of $S^n_3, \dots, S^n_{n-3}$?
- When can we choose a basis for $\mathrm{Tor}^S_\bullet(R,k)$ consisting of monomial cycles?

Few monomial ideals are known to admit such a basis when $n \geq 4$. Our theorems would tell us a lot about the algebra structure of $\mathrm{Tor}^S_\bullet(R,k)$. However, most known classes of such ideals are already known to be Golod by other means.

- When does the vanishing of products of monomial cycles imply I is Golod?

We were able to do this in four variables, but it doesn't hold in general for five variables or more, for two reasons. First, even if monomial cycle products vanish, other products may not, as the example $I = (x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_5^2)$

$$(\bar{x}_5 e_5) \wedge (\bar{x}_1 e_{234} - \bar{x}_2 e_{134}) = \overline{x_2 x_5} e_{1345} - \overline{x_1 x_5} e_{2345}$$

shows. Second, higher order products appear, and they may be non-zero, even if all other products are trivial [3]. It would be interesting to find classes of ideals for which we can answer this question positively.

- Can a similar classification be done for binomial, trinomial, …?

References

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